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A Simple Statement of Proof of Reciprocal-Theorem.

BY J. C. FIELDS.

From the Gaussian Criterion the proof of the Reciprocal-Theorem may be very briefly and directly evolved in the following manner:

Suppose p, q to be relatively prime odd integers and μ the number of least positive remainders (to mod. p) greater than $\frac{1}{2}p$, of the series

$$q, 2q, 3q, \dots, \frac{1}{2}(p-1)q. \quad (1)$$

Let ν be the like number when p and q are interchanged. We may separate (1) into groups, in each of which the members lie in value between two successive multiples of p , thus:

$$\left. \begin{aligned} & q, 2q, \dots, \left[\frac{p}{q}\right]q; \left(\left[\frac{p}{q}\right] + 1\right)q, \dots, \left[\frac{2p}{q}\right]q; \dots; \\ & \left(\left[\frac{sp}{q}\right] + 1\right)q, \dots, \left(\left[\frac{sp}{q}\right] + \kappa\right)q, \dots, \left[\frac{(s+1)p}{q}\right]q; \dots; \\ & \left(\left[\frac{tp}{q}\right] + 1\right)q, \dots; \frac{(p-1)q}{2}. \end{aligned} \right\} \quad (2)$$

where in the $(s+1)^{\text{th}}$ group the terms lie in value between sp and $(s+1)p$.

In this group any term $\left(\left[\frac{sp}{q}\right] + \kappa\right)q$ gives a least positive remainder $> \frac{1}{2}p$ if $\left(\left[\frac{sp}{q}\right] + \kappa\right)q - sp > \frac{1}{2}p$, i. e. if $\kappa > \frac{(2s+1)p}{2q} - \left[\frac{sp}{q}\right]$. The number of terms in the group satisfying this condition is of course the whole number of terms in the group less the greatest value of κ for which the condition is not satisfied, i. e.

$$\left(\left[\frac{(s+1)p}{q}\right] - \left[\frac{sp}{q}\right]\right) - \left(\left[\frac{(2s+1)p}{2q}\right] - \left[\frac{sp}{q}\right]\right) = \left[\frac{(s+1)p}{q}\right] - \left[\frac{(2s+1)p}{2q}\right].$$

Now in the $(t+1)^{\text{th}}$ and last group, or rather partial group, tp is the greatest multiple of p which is $< \frac{1}{2}(p-1)q$, and therefore $t = \left[\frac{(p-1)q}{2p}\right]$. Suppose

$p > q$, then $t = \frac{1}{2}(q-1)$, $\frac{1}{2}(p-1)q - tp = \frac{1}{2}(p-1)q - \frac{1}{2}(q-1)p < \frac{1}{2}p$.

The last group therefore gives no remainder $> \frac{1}{2}p$, for the last remainder $\frac{1}{2}(p-1)q - tp$, which is the greatest in the group, is $< \frac{1}{2}p$.

Summing up from all the groups the numbers of terms giving remainders $> \frac{1}{2}p$, we get

$$\begin{aligned} \mu &= \sum_s^{t-1} \left(\left[\frac{(s+1)p}{q} \right] - \left[\frac{(2s+1)p}{2q} \right] \right) = \sum_1^{\frac{1}{2}(q-1)} \left[\frac{sp}{q} \right] - \sum_1^{\frac{1}{2}(q-1)} \left[\frac{(2s-1)p}{2q} \right] \\ &= \sum_1^{\frac{1}{2}(q-1)} \left[\frac{sp}{q} \right] - \sum_1^{\frac{1}{2}(q-1)} \left[\frac{(q-2\sigma)p}{2q} \right] = \sum_1^{\frac{1}{2}(q-1)} \left[\frac{sp}{q} \right] - \sum_1^{\frac{1}{2}(q-1)} \left[\frac{p}{2} - \frac{sp}{q} \right] \\ &= -\frac{p-1}{2} \cdot \frac{q-1}{2} + \sum_1^{\frac{1}{2}(q-1)} \left(\left[\frac{sp}{q} \right] - \left[\frac{1}{2} - \frac{sp}{q} \right] \right) \\ &= -\frac{p-1}{2} \cdot \frac{q-1}{2} + \nu + 2 \sum_1^{\frac{1}{2}(q-1)} \left[\frac{sp}{q} \right], \end{aligned} \quad (3)$$

for $\left[\frac{sp}{q} \right] + \left[\frac{1}{2} - \frac{sp}{q} \right] = 0$ or -1 according as the least positive remainder obtained on dividing sp by q is $<$ or $> \frac{1}{2}q$, and in our notation the number of such remainders which are $> \frac{1}{2}q$ is ν . We have from (3) therefore

$$\mu + \nu \equiv \frac{p-1}{2} \cdot \frac{q-1}{2} \pmod{2}. \quad (4)$$

The above reasoning holds also when p, q are two odd numbers having a greatest common measure r , with the observation that under μ we have reckoned all the terms in (2) which are multiples of p , as giving remainder $p > \frac{1}{2}p$ and not remainder $0 < \frac{1}{2}p$. The number of such terms is that of values of s for which $sp = mq$, $m \nless \frac{1}{2}(p-1)$; here $s = \frac{mq}{p} = \frac{mr}{p} \cdot \frac{q}{r}$, and the number of such values of s is evidently $\left[\frac{(p-1)r}{2p} \right] = \frac{r-1}{2}$. If, then, in defining μ we reckon multiples of p as giving a remainder 0, we have simply to substitute $\mu + \frac{1}{2}(r-1)$ for μ in (3), whence

$$\frac{p-1}{2} \cdot \frac{q-1}{2} + \frac{r-1}{2} + \mu - \nu = 2 \sum_1^{\frac{1}{2}(q-1)} \left[\frac{sp}{q} \right]$$

and

$$\mu + \nu \equiv \frac{p-1}{2} \cdot \frac{q-1}{2} + \frac{r-1}{2} \pmod{2}.$$